*Task developed by Dylan Sherman, University of Sydney, 2018*

# STATE DA VINCI DECATHLON 2018

## CELEBRATING THE ACADEMIC GIFTS OF STUDENTS IN YEARS 5 & 6

# MATHEMATICS - SOLUTIONS

TEAM NUMBER _______________

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | TOTAL | RANK |
|---|---|---|---|---|---|---|---|---|---|---|---|
| /16 | /10 | /4 | /3 | /4 | /8 | /6 | /5 | /6 | /5 | /67 | |

# QUESTION 1: CICADA CYCLES (16 MARKS)

It has been found that cicadas hatch with a life cycle that lasts a length always equal to prime numbers, creating unexpected hatching years at the end of each cycle. Some species have 13 years while cicadas in Eastern United States have 17 years. Being prime, the number can't be divided evenly by a smaller number than itself except for 1.

(a) Most other animals, including the predators of Cicadas have life cycles of only 2-8 years. A predator has a life cycle of 4 years, another inspect prey has a life cycle of 6 years and a cicada prey has a life cycle of 7 years. **At the end of 2000** all three creatures completed their life cycle. Create a table and list the years that the next 6 cycles will end for each creature (6 marks)

| Predator | Prey 1 | Prey 2 |
|---|---|---|
| 2004 | 2006 | 2007 |
| 2008 | 2012 | 2014 |
| 2012 | 2018 | 2021 |
| 2016 | 2024 | 2028 |
| 2020 | 2030 | 2035 |
| 2024 | 2036 | 2042 |

1 mark for each correct ROW.

(b) In which years will there be a predator and prey concluding their cycle together (i.e. hatching)? (3 marks)

**20012, 2024, 2036**

(c) Using part (e) explain why cicadas have **evolved to have prime number** life cycles (2 marks)

Not divisible with other numbers so will rarely coincide with more common life cycles of predators (1 mark).

Therefore, there will be less years that they may be at risk of being eaten by predators (1 mark).

(d) A cicada has a life cycle of 13 years. A life cycle ends at the end of 1995. Will the cicada's life cycle end (i.e. they hatch) at the end of 2173? Show your working to explain (2 marks)

**NO (1 mark). 1 marks for working included to support answer.**

(e) A new insect has been found to have a life cycle of 421 years. Do you think this insect is a cicada? Explain. (3 marks)

**YES (1 mark) – it is a prime number (1 mark). 1 mark for working to prove it is a prime number.**

## QUESTION 2: UNSURPRISING SPONTANEITY (10 MARKS)

Below are a set of results from a test that recorded how many milliseconds it took for individuals of different ages to press a buzzer once a green screen turned red at a random unpredictable time.

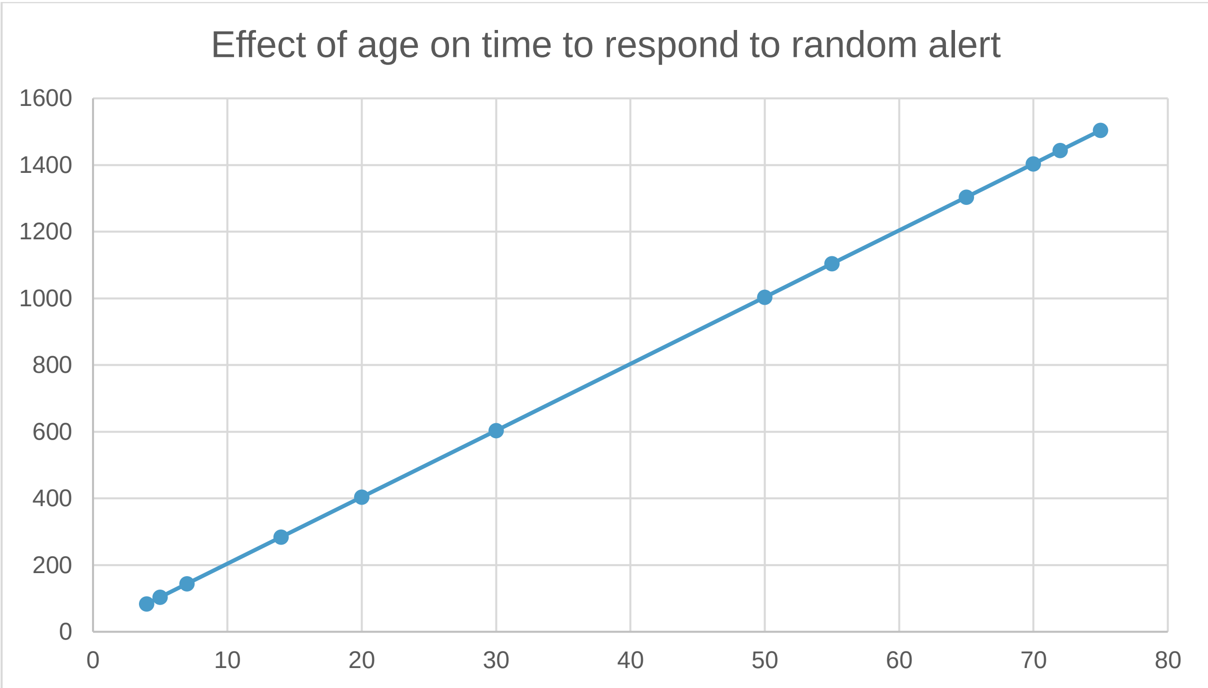
| Age | Time to respond (milliseconds) |
|---|---|
| 4 | 84 |
| 5 | 104 |
| 7 | 144 |
| 14 | 284 |
| 20 | 404 |
| 30 | 604 |
| 50 | 1004 |
| 55 | 1104 |
| 65 | 1304 |
| 70 | 1404 |
| 72 | 1444 |
| 75 | 1504 |

(a) Construct a line graph of this data. Consider including an appropriate title, axis scales and position of values on axis (age vs time to respond) (4 marks)

1 mark for title

2 mark for each axis labelled with units (Age (years) horizontal and time to respond (ms) vertical)

1 mark for appropriate scales




(b) By sketching a line that fits through the data, predict how long it would take for a 21-year-old to respond (2 marks)

See above (1 mark) prediction = 424 years (+/- 10 years). (1 mark)

(c) By extending the line in (b) predict how long it would take for a 90-year-old to respond (2 marks)

1904 (+/- 10 years) (1 mark) and 1 mark for extending the line

(d) Is it likely your answer in (c) would accurately reflect reality if a 90-year-old were to participate in the experiment? (2 marks)

No (1 mark) with reason (1 mark) e.g. difficult to see or slow muscles so may in fact be much slower than predicted – these effects tend to be amplified at older age and not simply linear.

## QUESTION 3: APPLE CORE (4 MARKS)

10 green and red apples are organised as below:

By switching only two adjacent apples at a time, what is the least number of switches you need to make so that the new arrangement below can be achieved? Explain.

Answer: 10 – **1 mark**

explanation with working – **3 marks**

Full 3 marks for showing that the results are the triangular numbers, i.e., two of each color requires 1 switch; three of each color requires 3 switches; 4 of each color requires 6 switches.

2 marks for a mathematical explanation involving patterns

1 mark for guess and check

# QUESTION 4: COOKIE THIEF (3 MARKS)

It is known that 2 cookies + 2 cookie jars = 1 cookie jar + 12 cookies. How many cookies are in a cookie jar?

10 cookies in each box **– full marks if working is shown** (may involve guess/check or algebra based working)

e.g.

2 cookie jars = 1 cookie jar + 10 cookies
1 cookie jar = 10 cookies.

# QUESTION 5: CLASSROOM CLOWNS (4 MARKS)

Three different teachers have students organised into groups for a group exercise. Mr James has four groups containing 2, 3, 4 and 5 students. Mr Smith has groups of 4, 5, 6 and 7 while Mrs Philips has groups of 6, 7, 8 and 9.

Each teacher must have the same number of students to supervise. Which **one** group should be moved to another teacher? Explain.

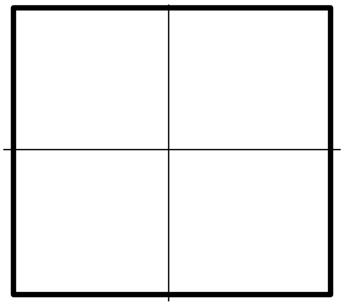
Mr. Jones has 14 students; Mrs. Smith has 22 students; Mrs. Philips has 30 student **(1 mark each)**

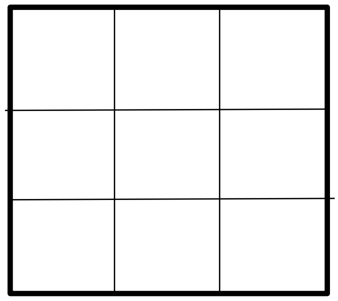
Move the group of 8 from Mrs. Philips to Mr. Jones **(1 mark)**

## QUESTION 6: FRUITY PROBLEM (8 MARKS)

(a) Alex orders apple juice boxes into square containers. If a container is divided with two dividers (each extend from one edge to the other of the square) he can fit 4 juice boxes. If he uses 4 dividers, he can fit 9 juice boxes **without each** touching each other. Draw a sketch to illustrate each situation (2 marks)

**1 mark each**





(b) How many juice boxes can fit (without touching) in a square create if 30 dividers are used? (3 marks)

| Numbers of divider | Number of juice boxes |
|---|---|
| 2 | 4 ($2^2$) |
| 4 | 9 ($3^2$) |
| 6 | 16 ($4^2$) |
| 8 | 25 ($5^2$) |
| 10 | 36 ($6^2$) |
| ... | ... |
| 30 | 256 |

The table can be continued up to 30 if the pattern is not obvious

**1 mark for 30 = 256; 2 marks for working**

(c) How many dividers are needed to hold 100 milk cartons? (3 marks)

One way to do this is to work backwards in the table. Since 100 = $10^2$, and 10 – 1 = 9, then 2 x 9 is the number of dividers that have to be used. So, 18 dividers.

**1 mark for answer; 2 marks for working.**

(d) If Alex wants to fit $b^2$ juice boxes into a square create (without touching) how many dividers are needed? (2 marks)

If we have $c^2$ cartons, then we can work from the right to the left in the table.

Corresponding to $c^2$ we have 2(c – 1) in the middle column. **(1 mark)**

S othe number of dividers = 2(c – 1). **(1 mark)**

# QUESTION 7: PLANNING TO WIN? (6 MARKS)

Hannah has a square crate that can hold 9 cartons of milk. She plays this game with Tane. First she puts a carton in, and then he does. They keep alternating in this way. However, the rule is that no three cartons can be in a line. The winner is the one who puts the last carton in the crate.

If Hannah always goes first, who should always win, Hannah or Tane?

Tane knows that the only way that he can win is if he forces the game to stop after 2, 4, or 6 cartons have been added to the crate. But from No-Three-In-A-Line Again, he knows that it has to be 4 or 6. So he is going to try to force the cartons (b) into one of the positions below, or a position that is symmetrical with one of these positions **(1 mark)**

| | | |
|---|---|---|
| b | b | |
| b | b | |
| | | |

| | | |
|---|---|---|
| b | | b |
| | | |
| b | | b |

| | | |
|---|---|---|
| b | b | |
| b | | b |
| | b | b |

On the other hand, Hannah will win if she can get the game to go to an odd number – 3 or 5. But she knows that 3 isn't possible. So she has to look for one of the 5 carton positions below **(1 mark)**

| | | |
|---|---|---|
| b | b | |
| | b | b |
| b | | |

| | | |
|---|---|---|
| b | b | |
| | | b |
| b | b | |

| | | |
|---|---|---|
| b | b | |
| | | b |
| b | | b |

This game can now be analysed using the fact that a 3 by 3 crate essentially has only three different kinds of squares: (i) a corner square; (ii) a middle square on a side; and (iii) the centre square. This means that Hannah only has three starting moves. But the centre square move is powerful. Although they both only have one winning end-game with a carton (b) in the centre square, Hannah is able to force the game to go her way. Watch **(1 mark)**

| | | |
|---|---|---|
| | | |
| | b | |
| | | |

| | | |
|---|---|---|
| | b | |
| | b | |
| | | |

| | | |
|---|---|---|
| | b | |
| | b | x |
| b | x | |

| | | |
|---|---|---|
| | | |
| | | |
| | | |

| | | |
|---|---|---|
| | | |
| | b | |
| | | |

| | | |
|---|---|---|
| b | | |
| | b | |
| | | |

| | | |
|---|---|---|
| b | | x |
| x | b | |
| b | | x |

| | | |
|---|---|---|
| | | |
| | | |
| | | |

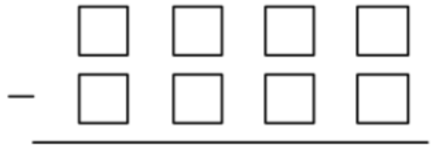
Hannah plays the centre square. Now, by symmetry, Tane has only two moves that he can make. In Game 1 above he plays in the middle square on an edge. In Game 2 he plays in the corner square **(1 mark)**

To win, Hannah only has to put in a carton so that Tane's winning position can't be achieved. In both cases she plays in the corner as shown. Crosses show where Tane can't play. Now wherever Tane plays, Hannah can finish the game by completing the crate as above. Hence Hannah can always win. **(2 marks)**

Please note: award a total of **four marks** for working and **two marks** for the correct answer (i.e. Hannah will always win). This means that teams who provide no working but write the correct answer will receive two marks.

## QUESTION 8: MINIMISING DAMAGE (5 MARKS)

Use the digits 0, 1, 2, 3, 4, 5, 6, 7 to find the smallest answer possible in the problem below:



$$\begin{array}{r} 4012 \\ -\ \underline{3765} \\ 247 \end{array}$$

**1/2 mark for each number, 1 mark for answer**

# QUESTION 9: SHAPE SHIFTING (6 MARKS)

Peter keeps a piece of string from a parcel that came for his birthday. It is 30 cm long.

He plays with it and makes different shapes all with the same perimeter of 30 cm.

Then he wonders about polygons.

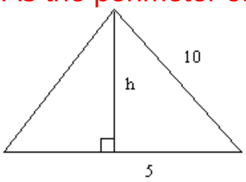
Which has the biggest area: an equilateral triangle, a square or a regular hexagon?

His sister Kiri says that she thinks he can get more area inside a circle with that perimeter.

Which of these figures has the biggest area?

Note: the area of a square = (side length)$^2$
area of a circle = π x (radius)$^2$ where the radius is the distance from the centre of the circle to an edge using a straight line.
the area of an equilateral triangle = $\frac{\sqrt{3}}{4} \times (side\ length)^2$

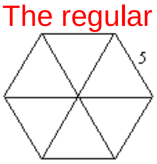
Equilateral triangle **(1 mark)**
As the perimeter of the equilateral triangle is 30 cm, one of its sides is 10 cm.




If we draw the perpendicular of the triangle, we see that, by Pythagoras' Theorem, 102 = 52 + h2. So, h2 = 102 – 52 = 100 – 25 = 75. So h = √75. The area of the equilateral triangle is, therefore, AT = ½ x 10 x √75. This is approximately 43.30 cm2.

Square **(1 mark)**
Since the square has four sides and perimeter 30 cm, one of its sides has length 7.5 cm. So its area is AS = 7.5 x 7.5 = 56.25.

Regular hexagon **(2 marks)**
The regular hexagon has six equal sides, so each side is of length 5 cm.




It is probably easiest to divide the hexagon into six equilateral triangles (1 mark) , each one of which has sidelength 5 cm. Using ideas from above, we see that the area of each of these triangles is ½ x 5 x √(52 - 2.5 2). This is approximately 10.83 cm2. So the area of the regular hexagon AH = 6 x 10.83 = 64.95 (1 mark)

Circle **(1 mark)**
If the perimeter of the circle is 30 cm, then we can find the radius using 30 = 2πr. So r = 30/2π = 15/2π. Leave this in this form for the moment. The area of the circle is then AC = πr2 = π(15/π)2 = 152/π = 225/π = 71.62.

It is clear that the circle has the biggest area. It's interesting that the areas of the regular polygons seem to increase as the number of sides increases. **(1 mark)**

## QUESTION 10: AGE OLD DILEMA (5 MARKS)

A Greek mathematician Diophantus developed the following problem:

> *When first the marriage knot was tied between my wife and me,*
> *Her age did mine as far exceed as three plus three does three;*
> *But when three years and half three years we man and wife had been*
> *Our ages were in ratio then as twelve is to thirteen.*

How old were they on their wedding day?

Method 1. Guess and Improve. Try an initial guess. If this doesn't work, then try another. Use the first guess to make the second one a better guess.

Suppose he was 30, then his wife was 33. After 3 + 3/2 = 4 1/2 years, he was 34 1/2, and his wife was 37 1/2. Now the ratio of his age to hers is 34 1/2: 37 1/2which equals 1:1.086 or 12:13.043. This is close but not close enough.

So try his age as 33. Then his wife's age was 36. After 4 1/2 years, he became 37 1/2, and she became 40 1/2. The ratio now is 1:1.08or 12:12.96. Since 12.96 is on the other side of 13 to 13.043, the next guess should be between 30 and 33.

So try 31.5 for Diophantus'age. His wife would then be 34.5. After 4 1/2 years, he is 36, and she is 39. The ratio between their ages is now 36:39, which equals 12:13. This is correct.

Method 2. Test every possible combination. The most efficient way to do this is using a computer program.

Method 3. Use algebra. Let his age be D. Then his wife's age is D + 3. An equation can be set up accordingly:

(D + 41/2)/[(D + 3) + 41/2] = 12/13.

Rearranging gives 13(D + 41/2) = 12[(D + 3) + 41/2].
'Tidying up' we have 13D + 58.5 = 12D + 90.
So D = 31.5,
as we found by Method 1.

Hence, when they got married, Diophantus was 31.5 and his wife 34.5.

**1 mark for wife's age, 1 mark for D's age. 3 marks for working (algebra method is awarded full 3 marks, guess and check only 2 marks)**