KNOX GRAMMAR SCHOOL

STATE

DA VINCI DECATHLON 2018

CELEBRATING THE ACADEMIC GIFTS OF STUDENTS IN YEARS 7 & 8

MATHEMATICS SOLUTIONS

TEAM NUMBER _____________

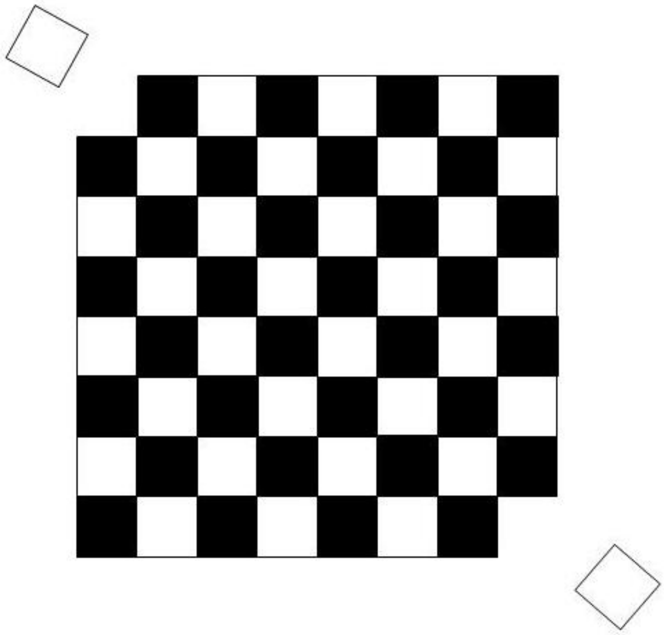
1	2	3	4	5	6	7	8	9	Total	Rank
/4	/6	/6	/4	/8	/4	/6	/4	/8	/50	

QUESTION ONE

THE MUTILATED CHESSBOARD

THE PROBLEM 4 MARKS

Mark is an avid board-gamer. His two favourite games are chess and dominoes. One day, he decides that he would like to combine the two and so he retrieves his chessboard and his 32 dominoes. By stroke of luck, each domino perfectly covers two squares of the chessboard. Therefore, his full collection covers the entire board. However, then he has the idea of **cutting off two corners**, as shown in the diagram below.



Mark **expects** that he will now be able to cover the board with 31 of his 32 dominoes. **Is this possible, and how?** Or, if it is not, **prove that it is impossible.** The dominoes must still cover two full squares each (i.e. **not diagonally**).

ANSWER SPACE

Is it possible? – NO (one mark for stating this)

Proof that it is impossible –

1. **Students should note that a domino, by covering two squares, must cover a black and white square (one mark)**

   Students should then deduce that because two white squares have been removed, there will inevitably be two black squares left on the board which cannot be filled by a domino, regardless of how they are positioned. Therefore, the board cannot be fully covered (two marks) (total of 4 marks).

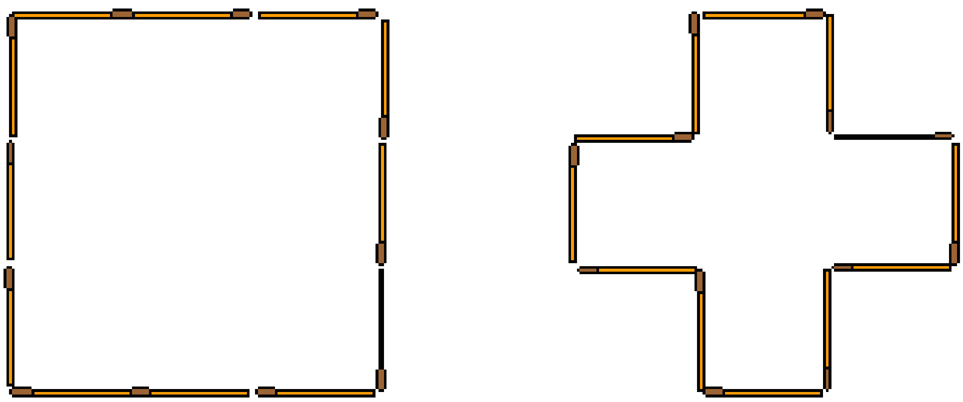
Note – students may try to prove that it is possible by having a domino hang off the board, only covering one square – this however would require more than 31 dominos to cover the whole board and is therefore not a correct answer.

QUESTION TWO

MATCH-ING GAME

THE PROBLEM 6 MARKS

John has twelve matches. Each is one unit in length. He creates polygons such as those below with ease, with areas of nine and five square units respectively. However, he doesn't **expect** to be able to create a polygon of only **four square units,** while using the entire lengths of all twelve matches as with these first two examples.



He does, though, persist, and is hit by a sudden rush of creative inspiration! He is able to create eight elegant polygons, all of four square units.

Your task is to draw three of them below, together with proof that the area of each of them is indeed four units square.

ANSWER SPACE

See following page for answers.

One mark for correct shape (note that they may be rotated) and one mark for sound proof of the area for each shape (this must be checked carefully).

i.e. 2 marks per shape, three shapes required for a total of 6 marks

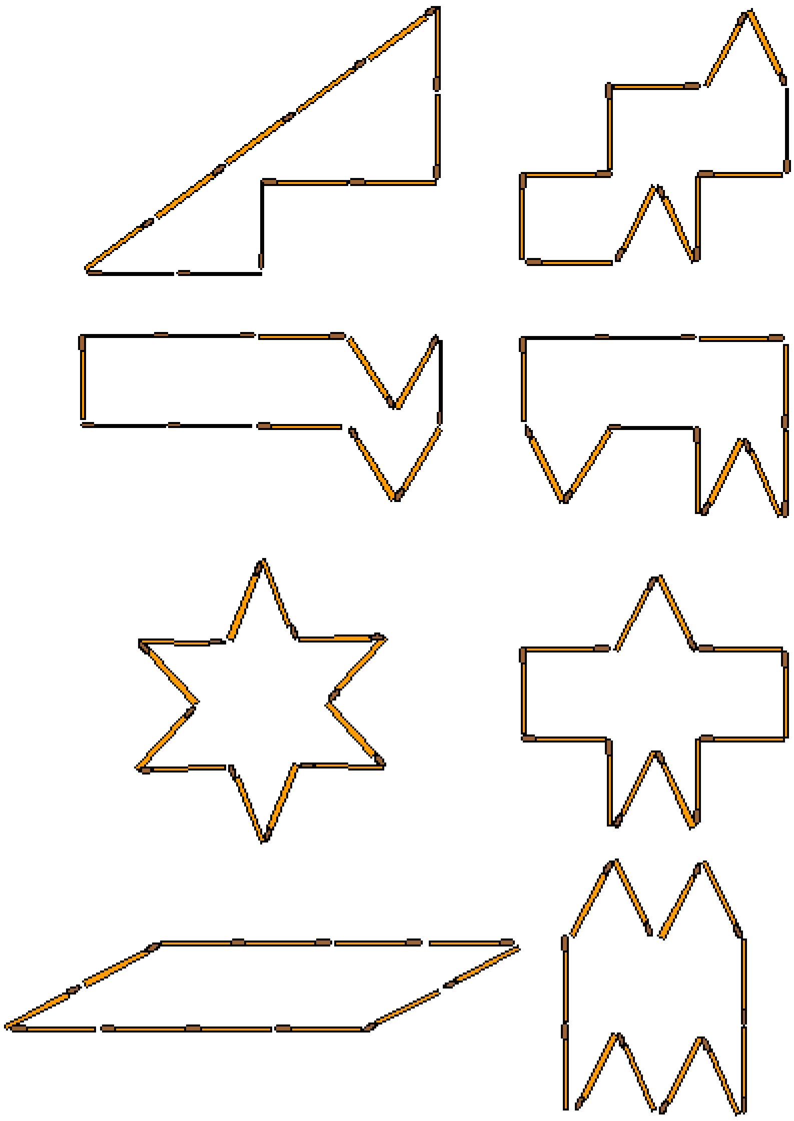
(1) Proof for the first shape (top left) is that by moving the three matches in the bottom right corner a triangle of 6 square units is formed. Working backwards, the three matches that are moved add 2 square units and therefore the original shape is 4 square units.

(2) Proof for most other shapes is simply that you have 3 full square units plus another that is split into ½ a square unit and two ¼ square units by the triangles.

(3) The star shape will only be four square units if the angles of the matches are set so that the base length of each triangle is 0.821 units. This must be noted to receive a mark for the proof if this shape is given as an answer.

NOTE – there are more correct alternatives in addition to the eight common answers provided below. Almost all will follow the same mathematical logic as (2) above, and will be similar in shape. Please award marks for any such correct alternative answers.

QUESTION TWO ANSWER SPACE CONTINUED



QUESTION THREE

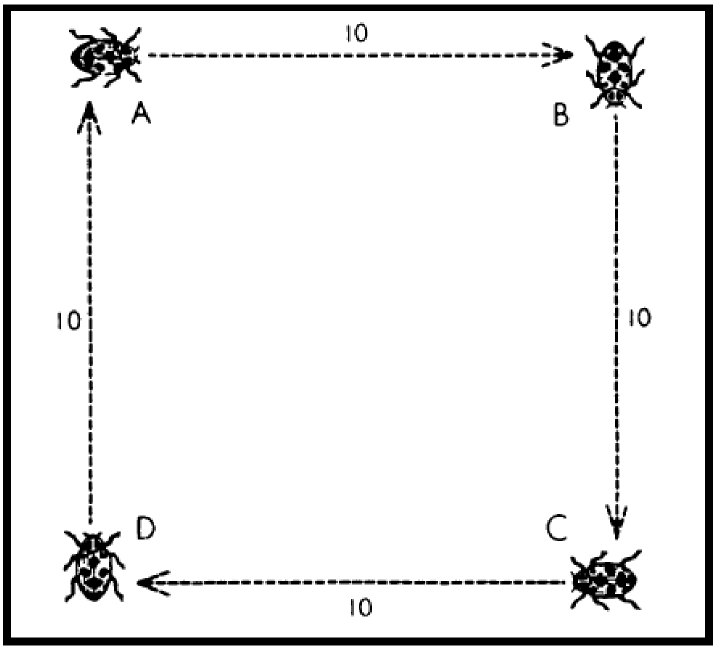
A BUG'S LIFE

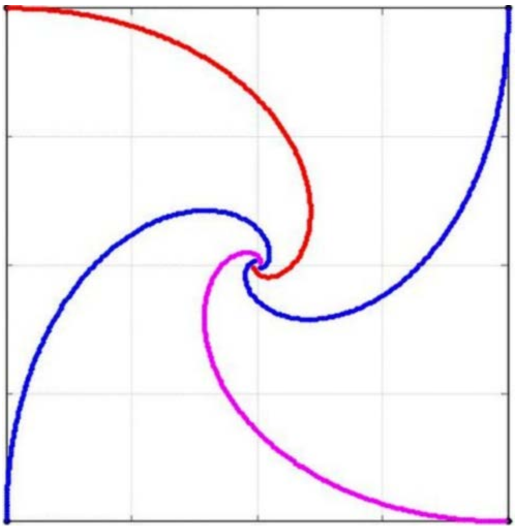
THE PROBLEM 6 MARKS

Four bugs occupy the corners of a square which is 10cm long on each side, as shown in the diagram below. The bugs are labelled A, B, C and D. Each bug is chasing the other, meaning that A is chasing B, B is chasing C, C is chasing D and D is chasing A. Every bug is crawling at the same speed.

Although these bugs are highly intelligent and can sense that they are travelling at the same speed as their target, they each do not know that their target is also chasing another bug. Therefore, **they do not expect to ever catch the bug in front of them,** because they anticipate only moving in a straight line.

Will each bug reach their target? If so, will they do so at the same time? **Finally, if so again, how far will each bug crawl until it reaches the one in front of it and why?**





ANSWER SPACE

Will they reach their targets? – YES – one mark

Will it be at the same time – YES – one mark

How far will each crawl – 10cm – one mark

Why? - Students should first note that by each bug chasing the one in front of them in a cycle, a spiral towards the centre of the square will form (one mark). At the same time, each bug will always be perpendicular to the one they pursue, and they are all moving at the same speed, meaning that the distance between each does not expand or diminish (two marks). It is always 10cm, and this will be the distance travelled by each once they all simultaneously reach the centre of the square (see picture above for visual explanation of this) (total of six marks).

QUESTION FOUR

DIVIDE, DIVIDE, REPEAT

THE PROBLEM 4 MARKS

A mathematician tells his class to write down a **three-digit number** (i.e. a number between 100 and 999) on a piece of paper. Just some of the numbers that the students choose include 394, 561, 777, 800 and 999. Then, the mathematician tells the class to duplicate their numbers (394394, 561561, etc.) before handing their paper to the person to their left.

The students are to **divide** the number they receive by 7, before passing it to the left again. Then, they **divide** the new number that they have on their desk by 11, and again pass the sheets of paper to their left. Finally, they are told to **divide** this fourth different number in front of them by 13. This time, however, each student is instructed to pass the sheet to their right **three times.**

To every student's amazement, the sheets in front of them now have their **original three-digit number on it. Explain how this is possible** for every three-digit number.

110	220	341	473	671
121	231	352	484	682
132	242	363	495	693
143	253	374	550	770
154	264	385	561	781
165	275	396	572	792
176	286	440	583	880
187	297	451	594	891
198	330	462	660	990

ANSWER SPACE

Students should firstly note (or it should be implicitly clear from their reasoning) that the passing of the sheets between students is not important. Each student receives their original sheet of paper back (one mark).

Students should then observe that duplicating a three-digit number (e.g. 529 becoming 529529) is the same as multiplying the original number by 1001 (one mark).

Now, 7x11x13 =1001, and these are the three numbers that each student's six-digit number was divided by (one mark). Therefore, each six-digit number is simply being divided by 1001, which will ALWAYS produce the original three-digit number (one mark).

(total of four marks).

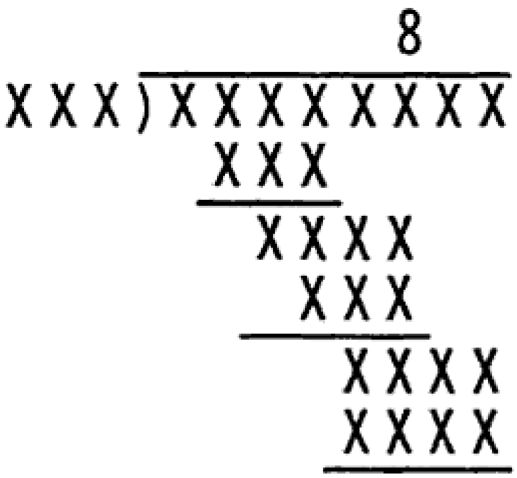
QUESTION FIVE

THE LONESOME 8

THE PROBLEM 8 MARKS

The following is a famous **long-division** problem, attributed to P. L. Chessin of the Westinghouse Electric Corporation. It was published in the April 1954 edition of *The American Mathematical Monthly* and remains the most popular problem ever published in that journal.

There are two ways to solve it. The first is trial and error, which would require testing the entirety of the **81 x 10^9** possible **permutations** of digits. The second is to realise that, through a few simple mathematical observations, the number of possible answers can quickly be reduced to (81 x 10^9)^0. So, without further ado, **solve** the puzzle!



ANSWER SPACE

The key is to observe that in two instances, two numbers are brought down instead of one (in the lines second and fourth from the bottom). In long division, this only occurs when there is a zero in the quotient. Therefore, the quotient is X080X (2 marks).

Then, it can be observed that 8 times the divisor produces a three-digit number (third line from the bottom), meaning 8 x divisor < 1000 or 8 x divisor ≤ 124 (1 mark).

But the last digit of the quotient times the divisor is four digits (i.e. more than 1000), and if the divisor is 124 or less than the last digit of the quotient can only be 9 (8x124 is not four digits). This also means that the bottom two lines are 1116 (9*124) (2 marks).

Now, the first subtraction in the division is XXXX – XXX = XX. This means that XXX = XXXX – XX ≥ 1000 – 99 = 901. However, 7 x 124 (the maximum divisor) is only 868, and therefore the first digit of the quotient is 8. The quotient is 80809 (1 mark).

From here it is fairly simple, as we have the quotient. 80809x123 is only a seven-digit number, whereas our dividend is eight digits. Therefore, the maximum divisor of 124 is in fact the divisor. (1 mark).

(p.t.o)

The problem therefore resolves as follows:

```
          80809
124 | 10020316
      992
      ----
       1003
        992
        ----
         1116
         1116
         ----
```

(1 final mark for all digits correct as above)

(total of 8 marks)

QUESTION FIVE ANSWER SPACE CONTINUED

QUESTION SIX

BILL SHOCK

THE PROBLEM

4 MARKS

Emily goes to the bank one day to open an account and deposit a cheque. However, the absent-minded bank teller, who is busy thinking about what to watch on Netflix this evening, switches the dollars and cents values when depositing Emily's money.

Later, Emily buys a 5-cent chocolate from the corner store, but is **shocked** when she checks her bank balance online! She now has **twice** as much as her original cheque.

What was the value of the cheque?

ANSWER SPACE

Let x be dollars and y be cents (both the actual amounts on the cheque, not the amount in the account). Therefore, Emily has 2x + 2y in her account after buying the chocolate (one mark).

Now, y has to be less than 100 (you can't have more than 99 cents), and from this we can form two options.

1. **If y is less than 50 cents, then 2x = y and 2y = x – 5 (because of the chocolate) (one mark).**
2. **If y is greater than 50 cents, then 2x + 1 = y and 2y – 100 = x – 5 (one mark)**

(this takes into account the fact that if y is greater than 50 cents, an extra dollar will be added when doubled).

We can then easily solve both sets of simultaneous equations. For set one, x = -5/3 which is negative and therefore not possible. For set two, x = 31 and y = 63.

Therefore, the cheque was for $31.63 (one mark).

($31.63x2 = $63.26, plus 0.05 = $63.31)

(total of 4 marks)

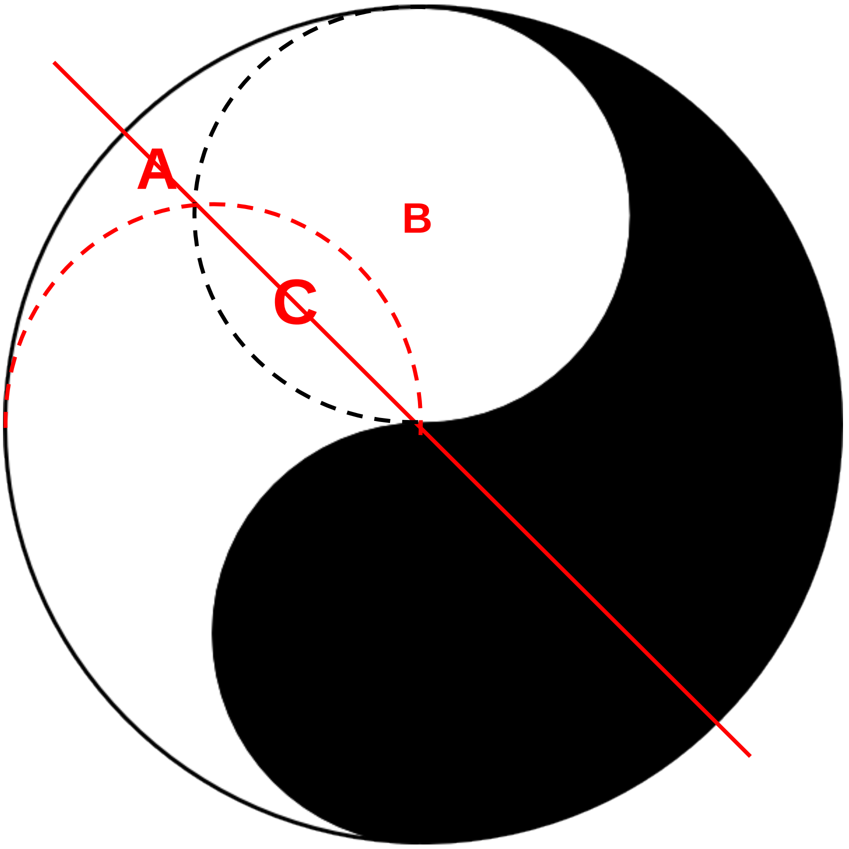
QUESTION SEVEN

YIN AND YANG

THE PROBLEM 6 MARKS

Two mathematicians were dining together at a Chinese restaurant when one noticed the Yin and Yang symbol on the restaurant's menu. He noted how attractive and proportional the symbol was, and then had a very interesting thought… **Is there a way to bisect both the entire symbol and also each of Yang and Yin (black and white respectively)?**

One curved line has been provided to you on the symbol below. Another line of exactly the same shape and length is also required, together with the line of bisection. **Draw these two lines and then use the diagram to prove that both Yin and Yang have been bisected.**



ANSWER SPACE

One mark for adding the other curved line (above in red).

One mark for adding the line of bisection ("…").

Proof:

The radius of circle B is half the radius of Yin and Yang. Using the area of a circle formula, if the radius of Yin and Yang is r, the total area of Yin and Yang is πr^2 and, and therefore the area of circle B is $\pi(r/2)^2$ which is $\pi r^2/4$. In other words, circle B is a quarter of the area of Yin and Yang (one mark).

(p.t.o)

If we then consider areas A, B and C as repeated around the whole symbol, we can say that the area of Yin and Yang (which we will call x) is as follows:

$X + 4C = 4A + 4B$ (one mark)

Now, $B = x/4$ (from above) and therefore $4B = x$

Therefore $4A = 4C$, $A = C$ and $A/2 = C/2$ (one mark)

Now, the bisecting line creates an area of $B + A/2 – C/2$ within Yin in the diagram, and because $A/2 = C/2$ the bisecting line creates an area equivalent to B.

(i.e. $B + A/2 – C/2 = B$)

Yin is 1/2 the area of Yin and Yang. 1/4 of Yin and Yang is equal to B. Therefore, the region $B + A/2 – C/2$ is also 1/4 of Yin and Yang and thus 1/2 of Yin alone. Thus, Yin (and in turn Yang) have been bisected (one mark).

(total of 6 marks)

QUESTION SEVEN ANSWER SPACE CONTINUED

QUESTION EIGHT

A SPIRITUAL JOURNEY

THE PROBLEM 4 MARKS

A Buddhist monk leaves to climb a mountain at sunrise one day. He follows a narrow, dirt track which spirals upwards towards the glittering temple at the summit. As he ascends, he stops many times and varies his speed greatly. He arrives at sunset.

After reaching the peak, he stays for many days of fasting and meditation. Then, one day at sunrise, he begins his descent. His speed is again variable and many stops occur. He reaches his original starting point at the base of the mountain by sunset.

Assume the times of sunrise and sunset do not change.

Will there be a point along the monk's route where he is located at exactly the same time on both the ascent and descent?

The answer to this may seem very simply, but the true answer is perhaps not what you expect at all. For one mark, state whether the answer to the question above is yes or no, and for three marks, prove your assertion!

ANSWER SPACE

One mark – YES, there will be such a point

Three marks – the best form of proof, which will score all three marks, is a graph such as the one on the following page. The shape of the lines representing the ascent and descent are not important, and information as to their gradient etc. is not provided in the question. Only the start and end points, which are the same for each, are important, because this means that the lines will always cross each other at a single point.

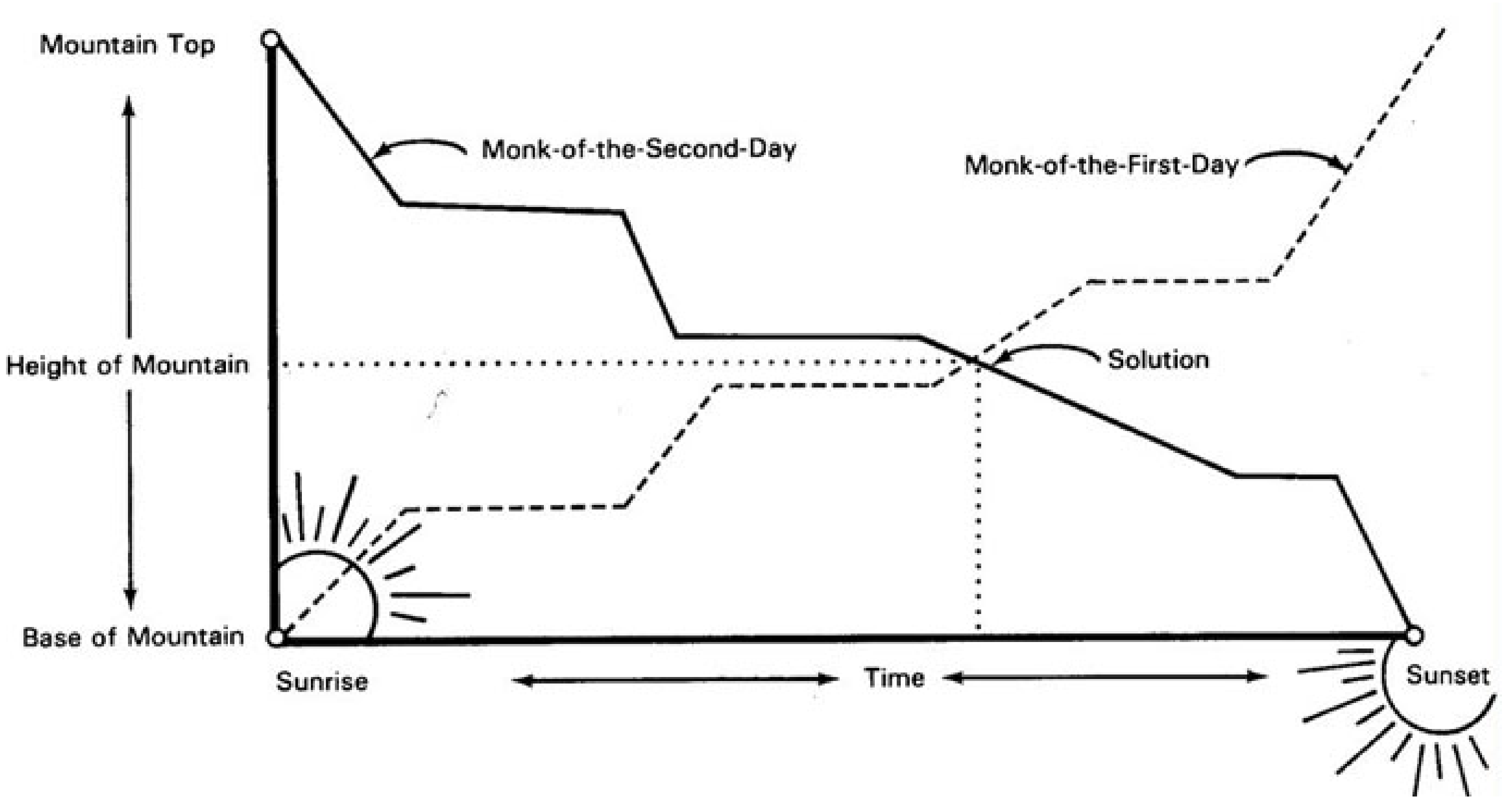
If graphed, marks are to be awarded for correct axes, start/end points and noting that the point of intersection represents a positive answer to the question.

Half marks can be awarded.

Other non-graphic explanations that show that there is indeed a fixed point should also be rewarded, if they correspond to what is represented in the graph. Use discretion in this regard.

(total of four marks)

QUESTION EIGHT ANSWER SPACE CONTINUED

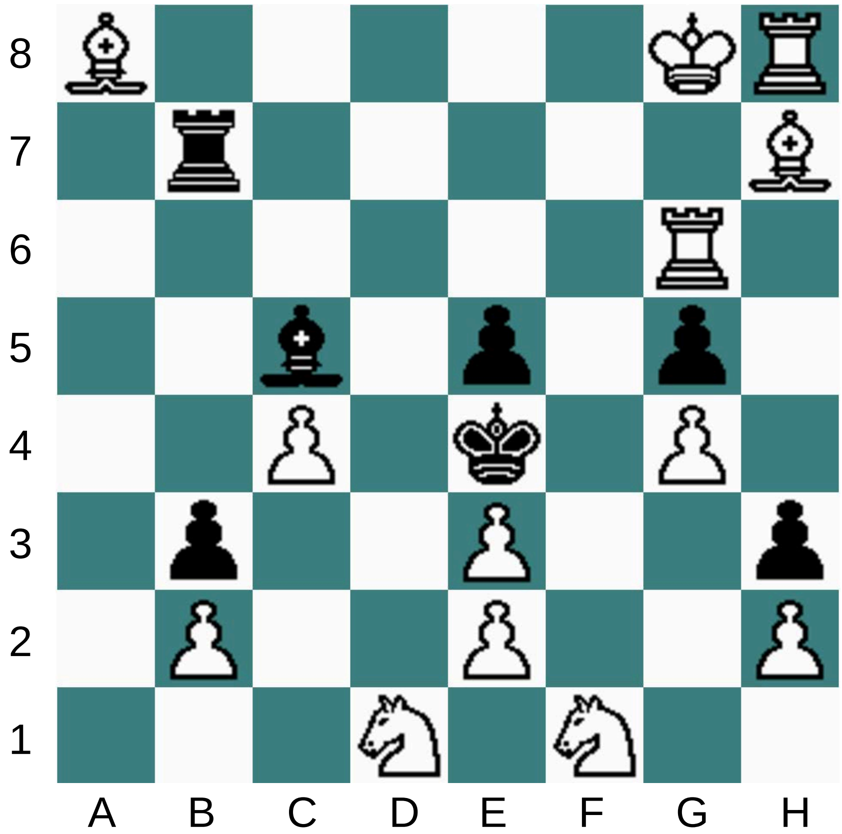


QUESTION NINE

UNCHECKMATE

THE PROBLEM 8 MARKS

This famous chess puzzle was devised by German problemist Karl Fabel. Your task is to find a move for **white** that will **NOT** result in checkmate of the black king. You will receive two marks for **identifying** the move and three marks for your **explanation**.



Many past viewers of this problem have complained that there is one element of the layout of pieces above which is **not possible.** While they are incorrect, it is understandable that such a complaint would be raised. For an **extra four marks,** can you **identify** the complaint and **explain** why it is incorrect?

ANSWER SPACE

The move: G6 to C6 (one mark)

Explanation: This move checks the black king via the white bishop at H7 (one mark). The black rook at B7 must then take this bishop, meaning the king is no longer in check (one mark) because the rook moved to C6 is now blocking the other bishop at A8 (one mark).

Bonus: The complaint is that two white bishops are on white squares, which is not possible (two marks). However, this is not true because a white pawn could and indeed must have made it to the end of the board on a white square and been swapped for a bishop (which although unusual is of course possible) (two marks). (total 8 marks)