KNOX
GRAMMAR
SCHOOL

STATE

DA VINCI DECATHLON 2018

CELEBRATING THE ACADEMIC GIFTS OF STUDENTS

IN YEARS 9, 10 & 11

MATHEMATICS SOLUTIONS

TEAM NUMBER ______________

1	2	3	4	5	6	7	8	9	Total	Rank
/4	/4	/7	/4	/6	/6	/5	/5	/9	/50	

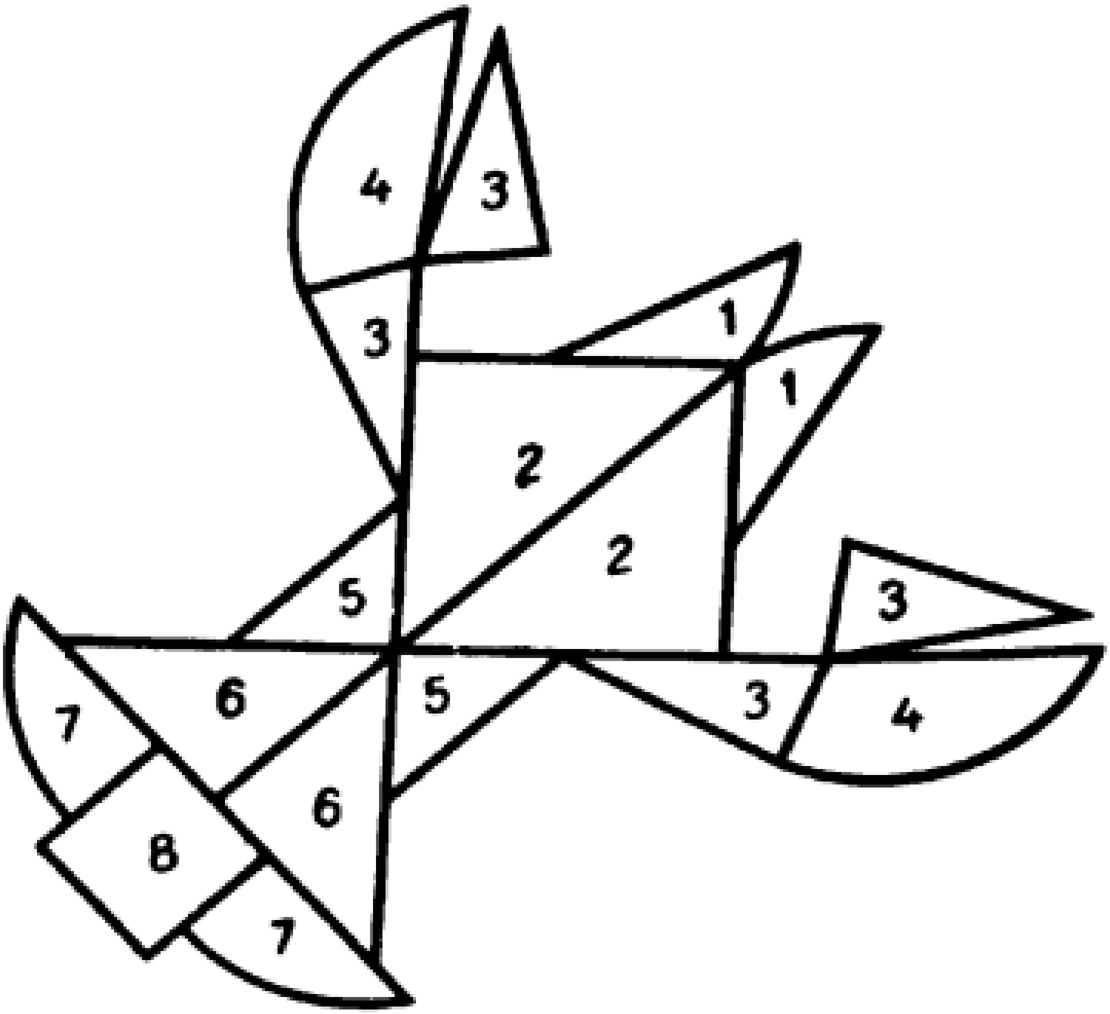
QUESTION ONE

ABSOLUTELY CRAY-ZY

THE PROBLEM 4 MARKS

Below is a **crayfish** made out of 17 numbered shapes. Your task is to rearrange these shapes, in the same proportions, to form a **square** and a **circle.** Make sure to write the numbers on each shape in your answers.

You do not have to cut out the shapes. Just redraw them in the answer space on the following page, making sure to measure the length in order to ensure proportionality. **Pencil** is highly recommended!



QUESTION ONE ANSWER SPACE

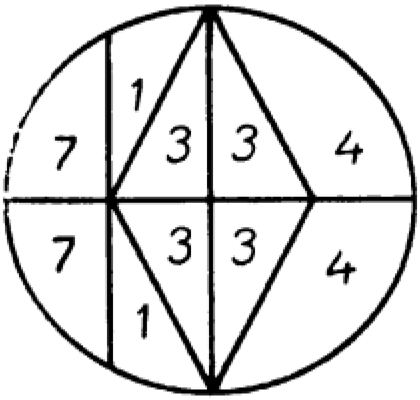
TWO marks for each fully correct per the diagrams below (total of four marks).

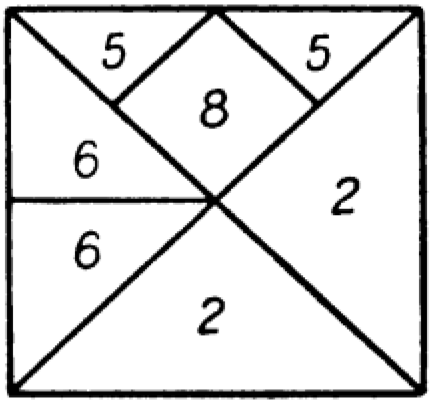
0/2 (per shape) – not attempted

0.5/2 – mostly incorrect (only one or two parts in correct position)

1/2 – roughly half correct

1.5/2 – mostly correct (only one or two parts in wrong position)





QUESTION TWO

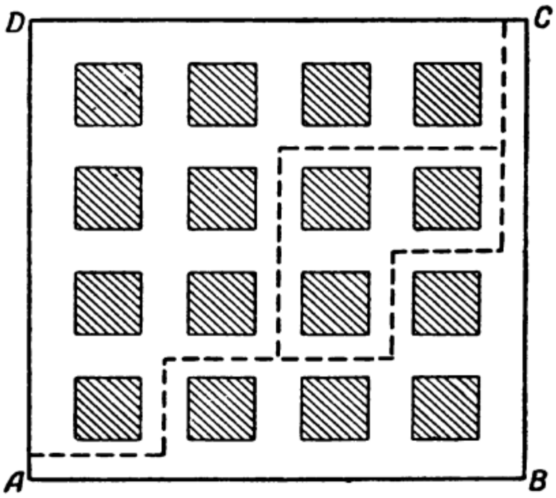
EN ROUTE

THE PROBLEM 4 MARKS

In a freshly designed city, there are 16 identical blocks. The area forms a perfect square. The **postman** for this new city needs to learn **every single possible route** from the post office, marked as A, to the city's exit, marked as C, so that no delivery trip is **unexpected.**

How many possible routes are there? Explain how you came to this answer.

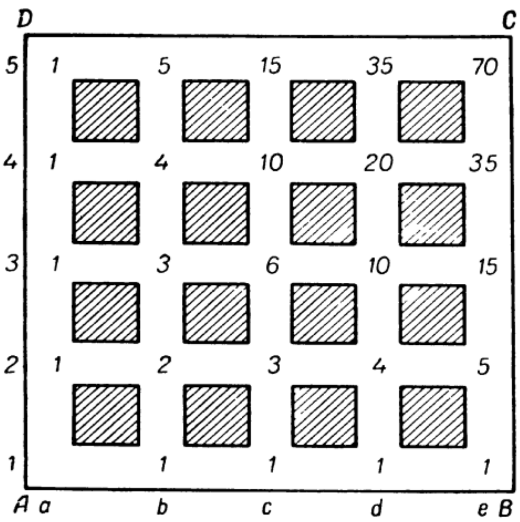
You are only to move upwards and right. Consider using a **diagram** for your explanation.



ANSWER SPACE

The answer is 70 (TWO MARKS FOR THIS ANSWER).

The remaining two marks are to be awarded for the following proof, or a similar valid proof.



The routes must be conceived through the different points which are to be passed through. These can be shown on the diagram to the right (it is not necessary for teams to provide a diagram, although it certainly helps).

The numbers on this diagram represent the number of different ways that each point can be reached. ONE MARK is to be awarded for teams that describe or exhibit this or a similar technique.

A pattern clearly emerges. Students should realise that the number of ways to reach a certain point is equal to the sum of two numbers – those immediately left and below the point (ONE MARK FOR POINTING THIS OUT). By then completing the pattern, the final point becomes 35 + 35 = 70 routes. (total of four marks)

QUESTION TWO ANSWER SPACE CONTINUED

QUESTION THREE

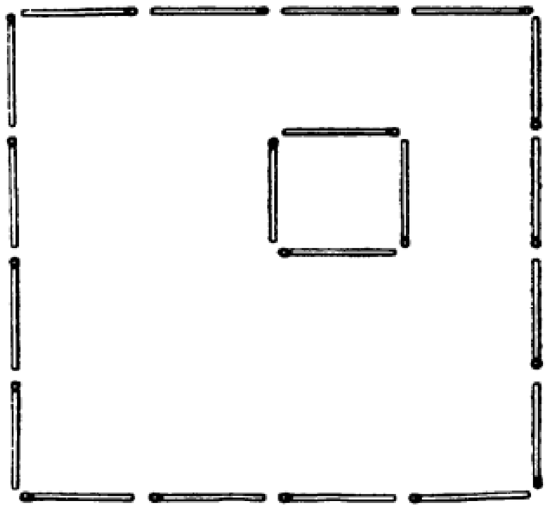
TOUGH MATCH

THE PROBLEM 7 MARKS

Match problems are among the most loved in the worlds of mathematics and puzzles, but also the most challenging. Can you solve these ones? At least one of the answers is not what you might **expect**… **SEE PAGE 7 FOR GENERAL ANSWER GUIDELINES**

Part One (1 mark) **ANSWER ON PAGE 7**

Using ten more matches, divide the space surrounding the square into five areas of identical size and shape. You can draw on the diagram provided.



Part Two (2 marks)

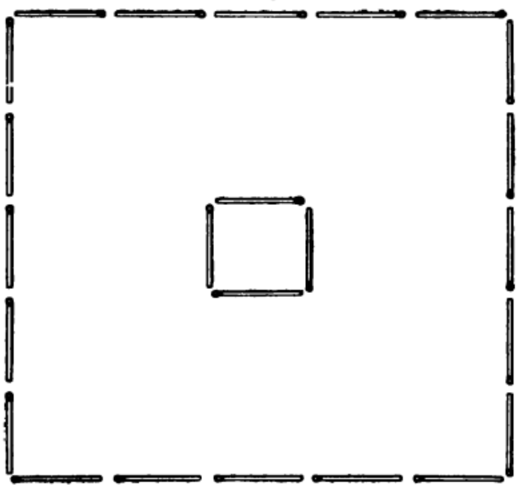
Now, do the same, but using: **ANSWER ON PAGE 7**

(a) 18 matches to form 6 identical areas; and

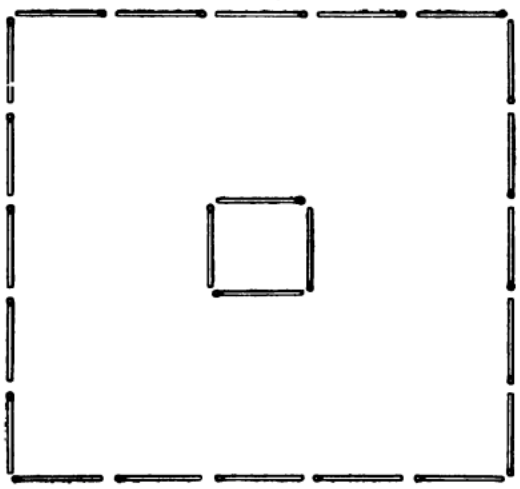
(b) 20 matches to form 8 identical areas.

Again, please use the two diagrams below for your answer.

a)



b)

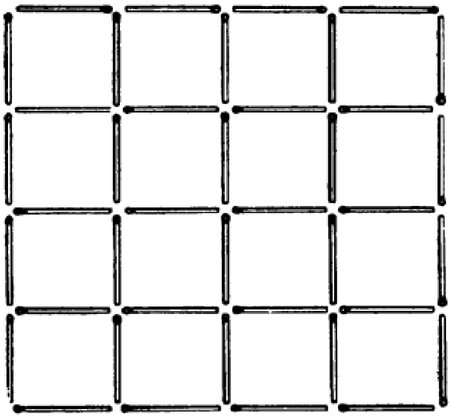


Part Three (1 mark)

What is the fewest number of matches to remove so that no squares of any size are left in the figure below?

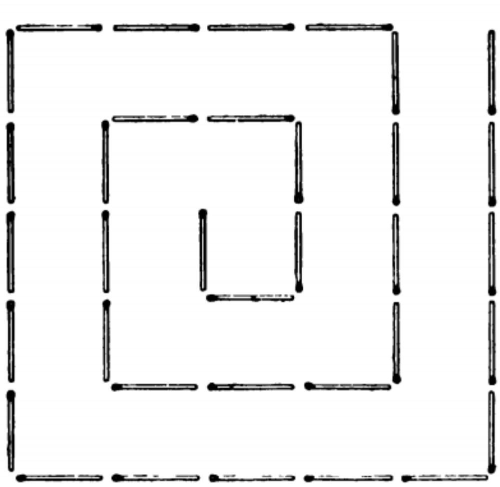
Answer: __________________

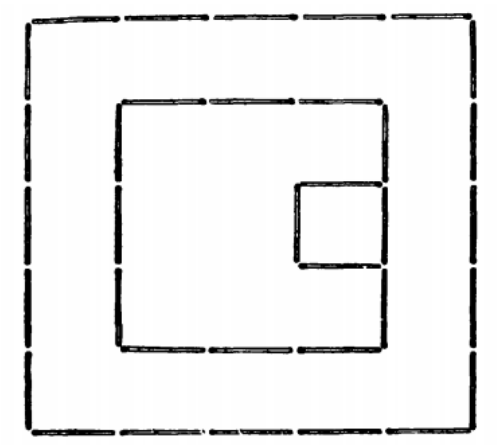
ANSWER: NINE



Part Four (1 mark)

Move four matches of this spiral to form an arrangement of three squares. You will need to redraw your answer to the right of the original figure. **ANSWER IMMEDIATELY BELOW**

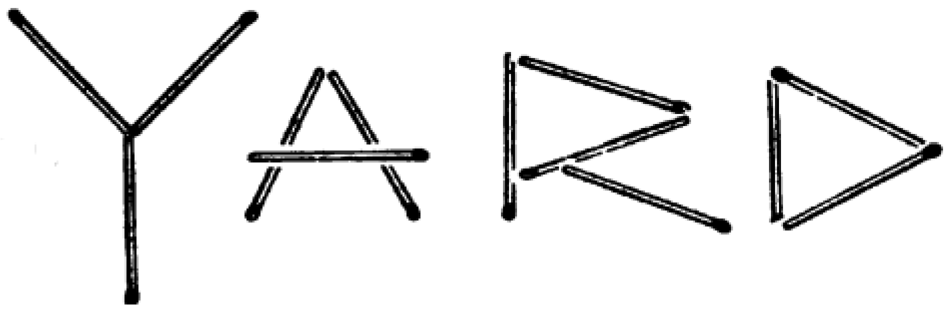




Part Five (2 marks)

Thirteen matches of two inches in length each can be put together to form one 'yard'. How?

Draw your answer below. **ANSWER IMMEDIATELY BELOW**

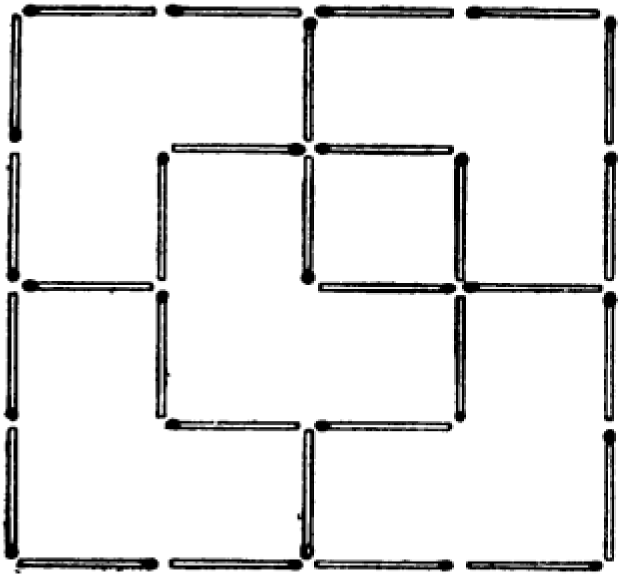


(Total of 7 marks)

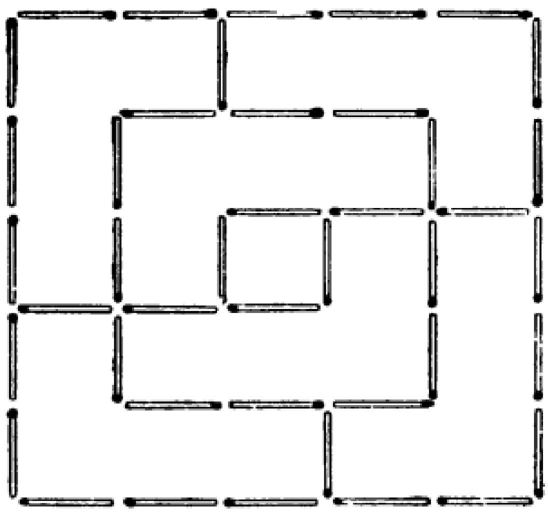
GENERALLY, answers to these questions are awarded full marks or zero marks – there are not to be half marks or 1/2, 2/3 etc. HOWEVER, please be careful when marking to ensure that answers that are rotated and look different to the diagrams provided in this answer booklet but are indeed correct are still awarded marks.

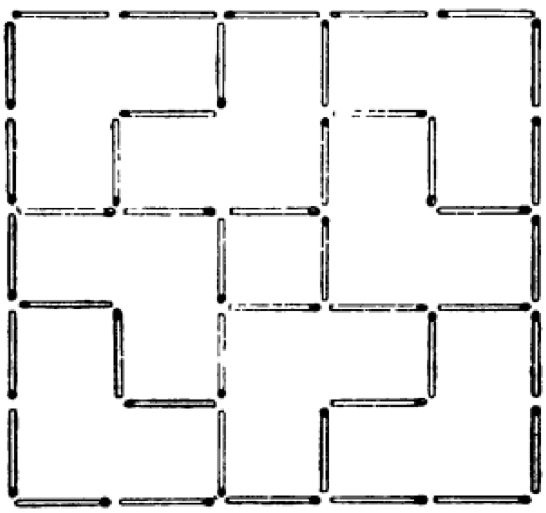
REMAINING ANSWERS:

QUESTION ONE



QUESTION TWO



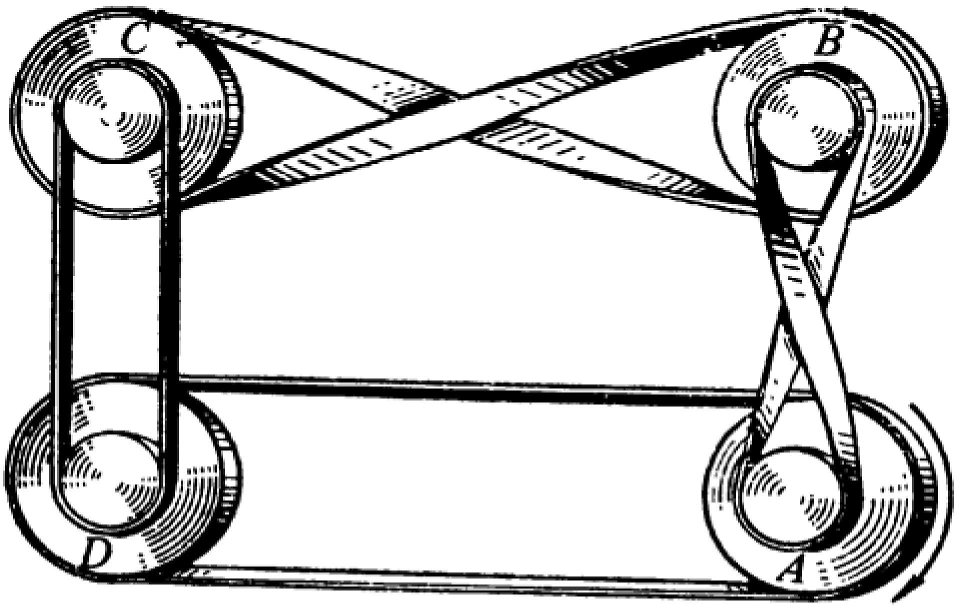


QUESTION FOUR

DIZZYING

THE PROBLEM 4 MARKS

Four belts marked A, B, C and D and connected as shown in the **diagram** below. A begins to rotate **clockwise** as indicated by the arrow. Use this information to answer the series of questions provided.



QUESTIONS

1. Will all four belts rotate based on the diagram above?

   ______________________________ **YES (ONE mark)**

2. Which way does each belt rotate (clockwise, anticlockwise or no rotation)?

   B - __________________________ **ANTICLOCKWISE** **0-1/3 = 0 marks**

   C - __________________________ **CLOCKWISE** **2/3 = ½ mark**

   D - __________________________ **CLOCKWISE** **3/3 = ONE mark**

3. If all four belts are crossed, like the one between B and C, will all still rotate?

   ______________________________ **YES (ONE mark)**

4. If the following numbers of belts are crossed, will all still rotate?

   1 - __________________________ **NO** **½ mark each, total of ONE mark**

   3 - __________________________ **NO** **(overalltotal of four marks)**

QUESTION FIVE

THINK INSIDE OF THE BOX

THE PROBLEM

6 MARKS

One cubical box houses **27** identical footballs. Another box of the same size contains **64** smaller footballs. The **boxes** are made of the same material, and both are filled to the top.

The layers of balls in each box are equal in number, and the outside layers touch the sides of each box. Fundamentally, however, there are **many more footballs** in one box, albeit they are a size smaller.

An employee working at Guerrilla Sports is tasked with determining the **weight** of the two boxes. The results surprise him!

(not an accurate diagram)

(1) Which is **heavier?** Show full, detailed proof.

(2) Is there a general rule for **cubic numbers** of footballs in boxes? If so, state it.

ANSWER SPACE

(1)

Students should first identify that the boxes are 3x3x3 and 4x4x4 respectively. As the boxes are the same size, this means that the diameters and radii of the larger balls are 4/3 the size of the smaller balls. (ONE mark)

The volume of a sphere formula must then be applied – V = $\frac{4}{3}\pi r^3$. (ONE mark for identifying this formula)

If V_1 is the smaller ball and V_2 the larger ball, and the radius of the smaller ball is x, the following can be determined:

$V_2 = \frac{4}{3} \times \pi \times (\frac{4}{3}x)^3$

$= \frac{256}{81}x^3\pi$

$V_1 = \frac{4}{3} \times \pi \times x^3$

$= \frac{4}{3}x^3\pi$

(1/2 mark for each correct volume, for a total of ONE mark)

Now, the total volume of the box with 27 balls is $27V_2$ and the total volume of the box with 64 balls is $64V_1$. Let these be B_2 and B_1 respectively.

(ONE mark for establishing this)

Therefore, $B_2 = 27 \times \frac{256}{81}x^3\pi$

$B_2 = \frac{256}{3}x^3\pi$

And, $B_1 = 64 \times \frac{4}{3}x^3\pi$

$B_1 = \frac{256}{3}x^3\pi$

Thus $B_1 = B_2$

To answer the question, neither box is heavier. They weigh the same. (ONE mark).

NOTE – there are other ways to answer this question by making deductions (i.e. without the full mathematical proof, particularly the B_1 and B_2 part above), however teams were asked to provide full, detailed proof. Those who do not should receive 1-3 marks out of f for part (1), depending on whether the other parts allocated marks above are provided.

(2) Students should be able to quickly deduce, perhaps through trial and error of one other pair of cubes, that the volumes of the boxes will always be the same. (ONE mark).

(total of six marks)

QUESTION FIVE ANSWER SPACE CONTINUED

QUESTION SIX

LOST FOR WORDS

THE PROBLEM 6 MARKS

This task is simple. Solve the following equations for the words below.

$$a^2 = bd \text{ and } ad = b^2c$$

B E

W O W

W H A T

W E I R D

S U D D E N

F E A R I N G

S U R P R I S E

U N P L A N N E D

U N E X P E C T E D

U N P R E D I C T E D

A D V E N T I T I O U S

U N A N T I C I P A T E D

R I D I C U L O U S N E S S

E X T R A O R D I N A R I L Y

ANSWER SPACE

Multiply the left and right sides of the equations, to form $a^3d = b^3dc$.

Therefore $a^3 = b^3c$, and, applying the cube root sign, $a = b\sqrt[3]{c}$. (ONE mark)

Now, students should deduce that if we are solving for the words, we are solving for the number of letters in each. Therefore, the pronumerals must be integers. We have words of 2-15 letters, and the only one of these which is a cube is 8.

So, $c = 8 = overcoat$. (ONE mark)

Therefore, $a = b\sqrt[3]{8}$, or $a = 2b$.

Substituting into the first given equation, $4b^2 = bd$ and so $4b = d$. (ONE mark)

Given we are only working with integers 2-15, b must therefore = 2 or 3, and d must equal 8 or 12.

8 has already been taken by c, and thus $d = 12 = mathematical$. (ONE mark)

Also, therefore, $b = 3 = oak$ (ONE mark), and $a = 6 = school$. (ONE mark)

(total 6 marks)

NOTE: only award marks IF the words are included for each pronumeral answer

QUESTION SEVEN

THE GREAT DIVIDE

THE PROBLEM 5 MARKS

Another short and sweet one. Prove than all **nine-digit** numbers whose three triplets of digits have a sum of the form **AAA** are divisible by **37.**

ANSWER SPACE

Let N be the nine-digit number. It can be written as:

$$N = 10^6 a + 10^3 b + c$$

The pronumerals above represent the three triplets of digits in the nine-digit number.

(ONE mark)

Now, students must realise at this point that ALL three digit numbers of the form AAA (i.e. 111, 222, …999) are divisible by 37.

(ONE mark for stating this at any point in the answer)

Therefore, if the three triplets of numbers from the nine-digit number sum to AAA, $a + b + c = 37k$.

(ONE mark)

Substituting for c, $N = 10^6 a + 10^3 b + 37k - a - b$

Therefore, $N = a(10^6 - 1) + b(10^3 - 1) + 37k$

(ONE mark)

$(10^6 - 1)$ and $(10^3 - 1)$ (i.e. 999,999 and 999) are also both divisible by 37, as is, of course, $37k$, and therefore all three terms of *N* are divisible by 37. Therefore, *N* is divisible by 37.

(ONE mark) (total of five marks)

(and thus any nine-digit number whose three triplets sum to AAA is divisible by 37)

QUESTION EIGHT

THE MAGIC NUMBER

THE PROBLEM 5 MARKS

You will be given a handful of **hints** below. Use them to answer the following **question**:

What is so unexpectedly special about the number 6174?

There are two parts to the answer.

HINTS

1. 

2. 0481, not 481.

3. Eventually.

4. a, b, c, d

   a, a, b, c

   a, a, b, b

   a, a, a, b

ANSWER SPACE

1. **6174 is the one four-digit number where the arrangement of its digits in descending order minus the arrangement of its digits in ascending order is itself.**

   (i.e. 7641 – 1467 = 6,174)

   (TWO marks for explaining this)

2. **What this means, furthermore is that any four-digit number which does not have four of the same digits (ONE mark for noting this), and where zero can be the first digit (either of the original number or when rearranged in ascending order) (ONE mark for noting this too) can be chosen, and by using the same formula as above (M – m), you will EVENTUALLY reach 6174 (ONE mark for this final explanation) (total of three marks for this part).**

   e.g. start with 4818. 8841 - 1488 = 7353. 7533 – 3357 = 4176. 7641 – 1467 = 6174.

   (total of five marks)

QUESTION NINE

GOING AROUND IN CIRCLES

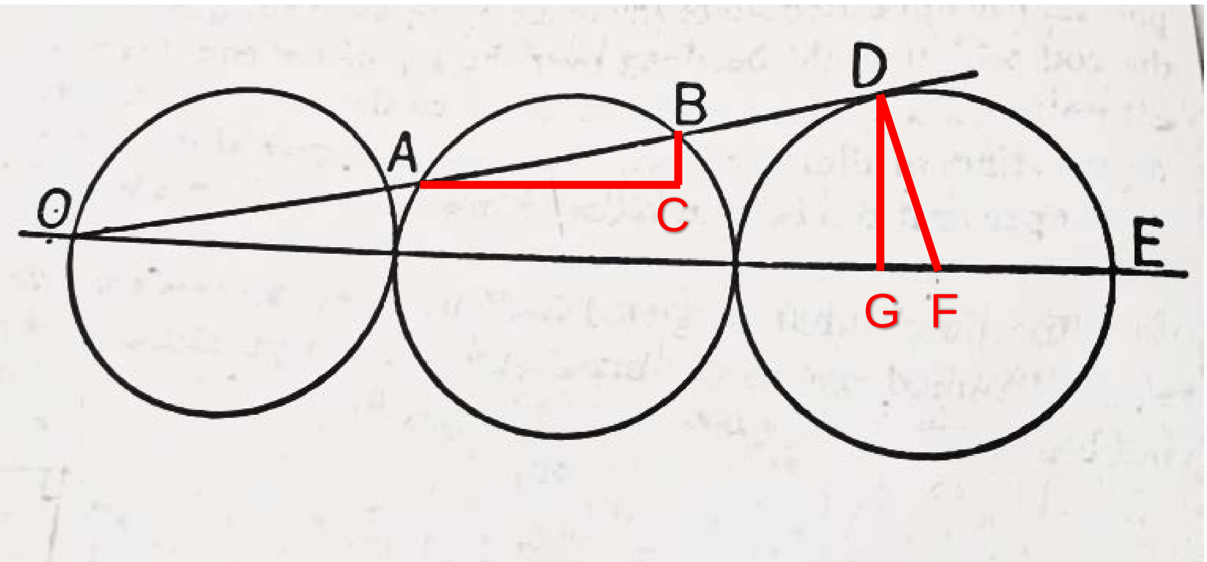
THE PROBLEM 9 MARKS

Three **circles** of equal radius, *r,* are drawn tangential to each other. Their centres all fall on the line OE. OD is a tangent of the right-most circle, extending to that circle from the outer intersection of OE and the left-most circle. AB is the chord in the middle circle formed by OD.

Determine the **length of AB** in terms of *r* (nine marks).

Some **construction** will be required.

DIAGRAM SPACE



ANSWER SPACE

Draw a perpendicular line from D to OE. Make the point G (see above). The equation of OD is therefore the ratio of DG to OG (i.e. y/x = DG/OG) (HALF mark)

Also draw a line from the tangent D to the centre of the right circle (F). Angle ODF is therefore a right angle (radius from tangent to centre rule) (HALF mark)

We now need to prove that triangles DGF and ODF are similar. We have a common angle, DFO, and both are right angled (DGF is a right angle as we have drawn DG to be perpendicular to OE, and we already know angle ODF is a right angle). We also have a common side, DF, and therefore the triangles are similar (AAS) (ONE mark).

So, using ratio of sides, OF/DF = OD/DG, and therefore DG = (OD x DF)/OF.

Now, DF = *r* and OF = 5*r.* Therefore, DG = OD/5 (ONE mark)

Using Pythagoras, $OD^2 = (5r)^2 - r^2$ and thus OD = $2r\sqrt{6}$

Therefore, DG = (2*r*$\sqrt{6}$)/5. (ONE mark)

Meanwhile, using ratio of sides in similar triangles again, OG/OD = DG/DF

Therefore OG = (2*r*$\sqrt{6}$ x (2*r*$\sqrt{6}$)/5)/*r*

So, OG = 24*r*/5 (ONE mark)

If we substitute these two values back into the equation for line OD, which was (from the first line above) y/x = DG/OG, we are left with:

y/x = ((2*r*$\sqrt{6}$)/5)/(24*r*/5)

Therefore, y/x = $\sqrt{6}/12$ (equation one) (ONE mark)

Now, we turn to the equation of the middle circle. This is (using basic circle equation rules, where O is the centre of the plane):

$(x - 3r)^2 + y^2 = r^2$ (equation two)

Solving equations one and two simultaneously, we are eventually left with:

$x = \frac{72r}{25} \pm \frac{8r}{25}\sqrt{6}$ (ONE mark)

These two potential x values represent the points A and B (as the intercepts of the equation of the middle circle and the equation of the tangent). Clearly, the difference between these values is $\frac{16r}{25}\sqrt{6}$, and this represents the length of AC (as drawn into the diagram on the previous page – students must do this).

By drawing AC we have constructed a right angled triangle. We can therefore find AB using Pythagoras ONCE we find the length of BC.

To do so, we must use $y/x = \frac{\sqrt{6}}{12}$ where x = AC.

This means $y = \frac{\sqrt{6}}{12} \times \frac{16r\sqrt{6}}{25}$ = 8r/25 (ONE mark)

Now using Pythagoras, $AB^2 = (8r/25)^2 + (\frac{16r\sqrt{6}}{25})^2$

This all boils down to an answer of AB = 8r/5! (ONE mark)

QUESTION NINE ANSWER SPACE CONTINUED